J Glob Optim (2007) 39:419–426
DOI 10.1007/s10898-007-9145-1

ORIGINAL PAPER

Global error bound for convex inclusion problems

Yiran He

Received: 7 July 2006 / Accepted: 29 January 2007 / Published online: 21 February 2007

Abstract The existence of global error bound for convex inclusion problems is discussed in this paper, including pointwise global error bound and uniform global error bound. The existence of uniform global error bound has been carefully studied in Burke and Tseng (SIAM J. Optim. 6(2), 265–282, 1996) which unifies and extends many existing results. Our results on the uniform global error bound (see Theorem 3.2) generalize Theorem 9 in Burke and Tseng (1996) by weakening the constraint qualification and by widening the varying range of the parameter. As an application, the existence of global error bound for convex multifunctions is also discussed.

Keywords Error bound · Convex inclusion · Convex multifunction

1 Introduction

The study on the existence of error bound is fruitful and has been developed in various directions by applying the theory of convex analysis and nonsmooth analysis, such as [3,4,6,8,11,13]. In this paper, we discuss the existence of global error bound for the solution set of the following convex inclusion problem:

$$Ax - b \in K, \tag{1}$$

where A is a continuous linear mapping from a normed linear space X to $\mathbb{R}^m$, $b \in \mathbb{R}^m$ a parameter, and $K \subset \mathbb{R}^m$ is a closed convex set. The problem (1) is said to have global error bound if there exists $\tau_b > 0$ (depending on the parameter b) such that

$$\operatorname{dist}(x, S_b) \le \tau_b \operatorname{dist}(Ax, K), \qquad \text{for all } x \in X, \tag{2}$$

Y. He (✉)
Department of Mathematics,
Sichuan Normal University, Chengdu, Sichuan 610068, China
e-mail: yiranhe@hotmail.com

where $S_b := \{x \in X : Ax - b \in K\}$ is the solution set of (1) and dist denotes the distance function induced by the norm.

In the setting of finite dimensional spaces, Hoffman [5] proved that if K is a polyhedral convex set then (1) has global error bound. This result was extended to infinite dimensional Banach spaces by Ioffe [7]. When K is not necessarily a polyhedral set, additional assumptions are needed to ensure the existence of global error bound. Reference [10] presented some equivalent characterizations on the existence of global error bound in terms of the subdifferential of the distance function from $Ax - b$ to K. Reference [1] proved that if

$$A(X) + K_\infty = \mathbb{R}^m, \tag{3}$$

where K_∞ denotes the recession cone of K, then global error bound (2) holds uniformly for all points b in the relative interior of $A(X) - K$, that is, τ_b is independent on b (see Theorem 9 therein). In this paper, we generalize Theorem 9 in Ref. [1] by weakening the constraint qualification condition (3) and by verifying that global error bound holds uniformly for all $b \in A(X) - K$; see Theorem 3.2. As an application, we address the existence of global error bound for convex multifunctions.

2 Preliminaries

For a normed linear space X, we use $\|\cdot\|$ to denote the norm on X, use $\mathbb{B}$ to denote the closed unit ball in X, use X^* to denote the dual space of X, and use $\langle\cdot,\cdot\rangle$ to denote the pairing between X and X^*. For any nonempty closed convex set $K \subset X$, we denote by σ_K the support function of K:

$$\sigma_K(x^*) := \sup_{x \in K} \langle x^*, x\rangle, \quad \forall x^* \in X^*,$$

$K^- := \{x^* \in X^* : \sigma_K(x^*) \le 0\}$, and $K^\perp := \{x^* \in X^* : \langle x^*, x\rangle = 0, \forall x \in K\}$. We use K_∞ and $\mathrm{barr}(K)$ to denote the recession cone and the barrier cone of K, respectively, i.e.,

$$K_\infty := \{d \in X : d + K \subset K\} \text{ and } \mathrm{barr}(K) := \{x^* \in X^* : \sigma_K(x^*) < \infty\}.$$

It is known that $(K_\infty)^-$ is the closure of $\mathrm{barr}(K)$ in the weak* topology. For any convex function $f: X \to \mathbb{R} \cup \{+\infty\}$, we use $\mathrm{epi} f$ to denote the epigraph of f and use f^* to denote the conjugate of f, i.e.,

$$\mathrm{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \le r\} \text{ and } f^*(x^*) := \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}.$$

For any continuous linear mapping A from X to $\mathbb{R}^m$, we denote by $A^*: \mathbb{R}^m \to X^*$ the adjoint of A and by $\ker(A^*)$ the kernel of A^*. For each nonempty set D in a finite dimensional space, $\overline{D}$ denotes the closure of D.

3 Error bound for convex inclusions

The following minimum norm duality theorem is well known; see [9,Theorem 5.13.1] and [14,Theorem 3.8.2] for its proof.

Lemma 3.1 *Let K be a nonempty convex set in a normed linear space X. Then*

$$\operatorname{dist}(x,K) = \max_{\|x^*\|\le 1}\left\{\langle x^*,x\rangle - \sigma_K(x^*)\right\}, \quad \textit{for all } x \in X.$$

Since every finite dimensional normed linear space is locally compact, the following lemma is an immediate consequence of Corollary 3.4 in Ref. [2].

Lemma 3.2 *Let T be a continuous linear mapping from a finite dimensional normed linear space Z to a linear topological space and let K be a nonempty closed convex set in Z. If $\ker(T) \cap K_\infty$ is a linear subspace of Z, then $T(K)$ is closed.*

Proposition 3.1 Suppose that A is a continuous linear mapping from a normed linear space X to $\mathbb{R}^m$, K is a closed convex set in $\mathbb{R}^m$, and $S := A^{-1}(K)$ is nonempty. If $\ker(A^*) \cap K^-$ is a linear subspace, then $\sigma_S = A^*\sigma_K$, where $(A^*\sigma_K)(x^*) := \inf\{\sigma_K(y^*) : A^*y^* = x^*\}$.

Proof Since $I_S(x) = (I_K \circ A)(x)$, in view of Theorem 2.3.1 (ix) in Ref. [14], we obtain that $(A^*\sigma_K)^*(x) = (\sigma_K)^*(Ax) = I_K(Ax) = I_S(x)$. Since σ_S is the conjugate function of I_S, it follows that σ_S is the weak* closure of $A^*\sigma_K$. It remains to prove that $\operatorname{epi}(A^*\sigma_K)$ is weak* closed. Let $B : \mathbb{R}^m \times \mathbb{R} \to X^* \times \mathbb{R}$ be a linear mapping defined by $B(y^*,r) = (A^*y^*,r)$. Then it is easy to prove that

$$B(\operatorname{epi}\sigma_K) \subset \operatorname{epi}(A^*\sigma_K) \subset \operatorname{cl}^* B(\operatorname{epi}\sigma_K), \tag{4}$$

where cl^* denotes the weak* closure.

Now we prove that $B(\operatorname{epi}\sigma_K)$ is weak* closed. It can be seen that $\operatorname{epi}(\sigma_K)$ is a weak* closed convex cone and hence

$$\begin{aligned}\ker(B) \cap (\operatorname{epi}\sigma_K)_\infty &= \ker(B) \cap \operatorname{epi}\sigma_K = (\ker(A^*) \times \{0\}) \cap \operatorname{epi}\sigma_K\\ &= (\ker(A^*) \cap K^-) \times \{0\}.\end{aligned}$$

Since $\ker(A^*) \cap K^-$ is a subspace, by virtue of Lemma 3.2, we obtain that $B(\operatorname{epi}\sigma_K)$ is weak* closed.

It follows from (4) that $\operatorname{epi}(A^*\sigma_K)$ is weak* closed. □

In what follows, some global error bound results will be proved.

Theorem 3.1 *Suppose that A is a continuous linear mapping from a normed linear space X to $\mathbb{R}^m$, K is a closed convex set in $\mathbb{R}^m$, and $S := A^{-1}(K)$ is nonempty. If*

$$\ker(A^*) \cap \overline{\operatorname{barr}(K)} = \ker(A^*) \cap K^\perp \tag{5}$$

then

$$\operatorname{dist}(x,S) \le \tau \operatorname{dist}(Ax,K), \quad \textit{for all } x \in X,$$

where

$$\tau := \sup\{\|y^*\| : y^* \in (A^*)^{-1}(\mathbb{B}) \cap \overline{\operatorname{barr}(K)} \cap T^\perp\} < \infty$$

and $T := \ker(A^*) \cap K^\perp$.

Proof Since $K^\perp \subset K^- \subset \operatorname{barr}(K)$,

$$T = \ker(A^*) \cap K^\perp \subset \ker(A^*) \cap K^- \subset \ker(A^*) \cap \overline{\operatorname{barr}(K)}.$$

This together with (5) implies that all the inclusions are equalities, so $\ker(A^*) \cap K^-$ is equal to T and is a linear subspace. It follows from Proposition 3.1 that

$$\sigma_S(x^*) = \inf\{\sigma_K(y^*) : A^*y^* = x^*\}.$$

This together with Lemma 3.1 yields that

$$\begin{aligned} \operatorname{dist}(x,S) &= \max_{\|x^*\|\le 1} \{\langle x^*, x\rangle + \sup_{A^*y^*=x^*} -\sigma_K(y^*)\} \\ &= \max_{\|x^*\|\le 1} \sup_{A^*y^*=x^*} \{\langle x^*, x\rangle - \sigma_K(y^*)\} \\ &= \max_{\|x^*\|\le 1} \sup_{A^*y^*=x^*} \{\langle y^*, Ax\rangle - \sigma_K(y^*)\} \\ &= \sup\{\langle y^*, Ax\rangle - \sigma_K(y^*) : y^* \in (A^*)^{-1}(\mathbb{B}) \cap \operatorname{barr}(K)\} \\ &\le \sup\{\langle y^*, Ax\rangle - \sigma_K(y^*) : y^* \in (A^*)^{-1}(\mathbb{B}) \cap \overline{\operatorname{barr}(K)}\}. \end{aligned} \tag{6}$$

Since the recession cone of $(A^*)^{-1}(\mathbb{B}) \cap \overline{\operatorname{barr}(K)}$ is equal to $\ker(A^*) \cap \overline{\operatorname{barr}(K)}$ which is the linear subspace $T \equiv \ker(A^*) \cap K^\perp$ in view of (5), we have

$$(A^*)^{-1}(\mathbb{B}) \cap \overline{\operatorname{barr}(K)} = (A^*)^{-1}(\mathbb{B}) \cap \overline{\operatorname{barr}(K)} \cap T^\perp + T. \tag{7}$$

Since the recession cone of the set $(A^*)^{-1}(\mathbb{B}) \cap \overline{\operatorname{barr}(K)} \cap T^\perp$ is the set $\ker(A^*) \cap \overline{\operatorname{barr}(K)} \cap T^\perp$ which is equal to $T \cap T^\perp = \{0\}$, we have the set $(A^*)^{-1}(\mathbb{B}) \cap \overline{\operatorname{barr}(K)} \cap T^\perp$ is a bounded closed convex set. By the definition of τ, $\tau < \infty$ and

$$(A^*)^{-1}(\mathbb{B}) \cap \overline{\operatorname{barr}(K)} \cap T^\perp \subset \tau\mathbb{B}.$$

In view of (7),

$$(A^*)^{-1}(\mathbb{B}) \cap \overline{\operatorname{barr}(K)} \subset \tau\mathbb{B} + T.$$

It follows from (6) that

$$\begin{aligned} \operatorname{dist}(x,S) &\le \sup\{\langle y^*, Ax\rangle - \sigma_K(y^*) : y^* \in (A^*)^{-1}(\mathbb{B}) \cap \overline{\operatorname{barr}(K)}\} \\ &\le \sup\{\langle y^*, Ax\rangle - \sigma_K(y^*) : y^* \in \tau\mathbb{B} + T\} \\ &= \sup\{\langle u^* + v^*, Ax\rangle - \sigma_K(u^* + v^*) : u^* \in \tau\mathbb{B}, v^* \in T\} \\ &= \sup\{\langle u^*, Ax\rangle - \sigma_K(u^*) : u^* \in \tau\mathbb{B}\} \\ &= \tau \sup\{\langle u^*, Ax\rangle - \sigma_K(u^*) : u^* \in \mathbb{B}\} \\ &= \tau \operatorname{dist}(Ax, K), \end{aligned}$$

where the last equality follows from Lemma 3.1. □

Remark 3.1 In the proof of the above theorem, we show that the closed convex set $(A^*)^{-1}(\mathbb{B}) \cap \overline{\operatorname{barr}(K)} \cap T^\perp$ is bounded. Therefore, Corollaries 18.5.1 and 32.3.1 in Ref. [12] imply that the supremum in the definition of τ is actually attained at some extreme point of the set $(A^*)^{-1}(\mathbb{B}) \cap \overline{\operatorname{barr}(K)} \cap T^\perp$.

Corollary 3.1 *Theorem 3.1 holds if the condition (5) is replaced by the following one:*

$$\ker(A^*) \cap \overline{\operatorname{barr}(K)} = \ker(A^*) \cap (K - K)^\perp.$$

Proof Since $S = A^{-1}(K)$ is nonempty, let $x_0 \in X$ be such that $Ax_0 \in K$. Set $b_0 := Ax_0$ and $K_0 := K - b_0$. Then $\mathrm{barr}(K) = \mathrm{barr}(K_0)$ and $(K_0)^\perp = (K - K)^\perp$. Hence the condition (5) in Theorem 3.1 is satisfied with K replaced by K_0. Moreover, $S_0 := A^{-1}(K_0) = S - x_0$ is nonempty as so is S. Taking τ as defined in Theorem 3.1 and applying Theorem 3.1, one has

$$\mathrm{dist}(y + x_0, S) = \mathrm{dist}(y, S_0) \le \tau\, \mathrm{dist}(Ay, K_0) = \tau\, \mathrm{dist}(A(y + x_0), K), \quad \forall y \in X.$$

For every $x \in X$, taking $y = x - x_0$, we obtain that

$$\mathrm{dist}(x, S) \le \tau\, \mathrm{dist}(Ax, K).$$

This completes the proof. □

Theorem 3.2 *Suppose that A is a continuous linear mapping from a normed linear space X to $\mathbb{R}^m$ and K is a closed convex set in $\mathbb{R}^m$. If*

$$\ker(A^*) \cap \overline{\mathrm{barr}(K)} = \ker(A^*) \cap (K - K)^\perp \equiv L \tag{8}$$

then for all $b \in A(X) - K$,

$$\mathrm{dist}(x, S_b) \le \gamma\, \mathrm{dist}(Ax - b, K) \quad \textit{for all } x \in X,$$

where $S_b := A^{-1}(K + b)$ and

$$\gamma := \sup\{\|y^*\| : y^* \in (A^*)^{-1}(\mathbb{B}) \cap \overline{\mathrm{barr}(K)} \cap L^\perp\} < \infty.$$

Proof With the set K in Corollary 3.1 replaced by $K + b$, the conclusion follows immediately from Corollary 3.1. □

Remark 3.2 In Theorem 3.2, the constant γ is independent on b.

Corollary 3.2 *Besides those assumptions in Theorem 3.2, if K is a closed convex cone and*

$$\ker(A^*) \cap K^- = \ker(A^*) \cap K^\perp \tag{9}$$

then for every $b \in A(X) - K$,

$$\mathrm{dist}(x, S_b) \le \gamma\, \mathrm{dist}(Ax - b, K) \quad \textit{for all } x \in X,$$

where $\gamma := \sup\{\|y^*\| : y^* \in (A^*)^{-1}(\mathbb{B}) \cap K^- \cap L^\perp\} < \infty$ and $L = \ker(A^*) \cap K^\perp$.

Proof Since K is a closed convex cone, $\mathrm{barr}(K) = K^-$ and $(K - K)^\perp = K^\perp$. The former implies that $\mathrm{barr}(K) = \overline{\mathrm{barr}(K)}$. Thus, the assumption (9) implies that (8) is satisfied, and hence the conclusion follows from Theorem 3.2. □

The following result is a corollary of Theorem 3.2 and improves Theorem 9 in Ref. [1].

Corollary 3.3 *Suppose that*

$$A(X) + K_\infty = \mathbb{R}^m. \tag{10}$$

Then $\gamma := \sup\{\|y^*\| : y^* \in (A^*)^{-1}(\mathbb{B}) \cap \mathrm{barr}(K)\} < \infty$ and for all $b \in A(X) - K$,

$$\mathrm{dist}(x, S_b) \le \gamma\, \mathrm{dist}(Ax - b, K) \quad \textit{for all } x \in X. \tag{11}$$

Proof Since $(K_\infty)^- = \overline{\mathrm{barr}(K)}$, (10) implies that $\ker(A^*) \cap \overline{\mathrm{barr}(K)} = \{0\}$. Since $(K - K)^\perp \subset \mathrm{barr}(K)$, it follows that

$$0 \in \ker(A^*) \cap (K - K)^\perp \subset \ker(A^*) \cap \overline{\mathrm{barr}(K)} = \{0\}.$$

Therefore $\ker(A^*) \cap \overline{\mathrm{barr}(K)} = \ker(A^*) \cap (K - K)^\perp = \{0\}$, i.e., (8) holds. In view of Theorem 3.2, for all $b \in A(X) - K$,

$$\mathrm{dist}(x, S_b) \le \gamma_1 \, \mathrm{dist}(Ax - b, K) \quad \text{for all } x \in X,$$

where $\gamma_1 := \sup\{\|y^*\| : y^* \in (A^*)^{-1}(\mathbb{B}) \cap \overline{\mathrm{barr}(K)}\}$.

Since the origin belongs to $\mathrm{barr}(K)$ and is an interior point of $(A^*)^{-1}(\mathbb{B})$, it follows from Exercise 1.4 in Ref. [14] that

$$\overline{(A^*)^{-1}(\mathbb{B}) \cap \mathrm{barr}(K)} = \overline{(A^*)^{-1}(\mathbb{B})} \cap \overline{\mathrm{barr}(K)} = (A^*)^{-1}(\mathbb{B}) \cap \overline{\mathrm{barr}(K)}.$$

So $\gamma_1 := \sup\{\|y^*\| : y^* \in \overline{(A^*)^{-1}(\mathbb{B}) \cap \mathrm{barr}(K)}\}$. By virtue of the continuity of the norm, we obtain that $\gamma = \gamma_1$. □

Remark 3.3 The main difference between Theorem 9 in Ref. [1] and Corollary 3.3 is that the former verified that global error bound (11) holds for those b in the relative interior of $A(X) - K$, while the latter verifies (11) holds for all $b \in A(X) - K$. Under the assumption (10), the scalar γ defined in Corollary 3.3 is equal to the scalar μ_5 defined in Ref. [1, p. 281]. In fact, (10) implies that $\ker(A^*) \cap \mathrm{barr}(K) = \{0\}$, so the set W_3 defined in Ref. [1] is equal to $\mathrm{barr}(K)$. This shows that $\gamma = \mu_5$.

We present an example to show that Theorem 9 in Ref. [1] is not applicable while Theorem 3.2 is applicable.

Example 3.1 Let $X = \mathbb{R}^3$, $K := \{y \in \mathbb{R}^3 : y_1 > 0, y_1 y_2 \ge 1, y_3 = 0\}$, and $A(z)$ is the projector of $z \in X$ onto the subspace $\{x \in \mathbb{R}^3 : x_3 = 0\}$. Then $\ker(A^*) = \{y \in \mathbb{R}^3 : y_1 = 0 = y_2\}$, $\mathrm{barr}(K) = \{y \in \mathbb{R}^3 : y_1 \le 0, y_2 \le 0\}$, and $(K - K)^\perp = \{y \in \mathbb{R}^3 : y_1 = y_2 = 0\}$. Therefore

$$\ker(A^*) \cap \overline{\mathrm{barr}(K)} = \{0\} \times \{0\} \times \mathbb{R} = \ker(A^*) \cap (K - K)^\perp,$$

that is, (8) holds and hence Theorem 3.2 is applicable.

But the conditions of Ref. [1,Theorem 9] are not satisfied, because neither K is a polyhedral convex set nor $A(X) + K_\infty = \mathbb{R}^m$.

Recall that γ is the constant in the conclusion of Theorem 3.2. By calculation, if we assume $\mathbb{R}^3$ with the ℓ_2-norm, then

$$\gamma = 1 = \sup_{x \notin S_b} \frac{\mathrm{dist}(x, S_b)}{\mathrm{dist}(Ax - b, K)}, \qquad \text{for every } b \in A(X) - K,$$

that is to say, in this example, γ is the smallest constant such that the conclusion of Theorem 3.2 holds. However, it is not known whether this observation is true for general problems.

4 Applications to convex multifunctions

Let $\Gamma: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multifunction. We use $\mathrm{Gr}(\Gamma)$ and $\mathrm{dom}(\Gamma) := \{x \in \mathbb{R}^n : \Gamma(x) \neq \emptyset\}$ to denote the graph and the domain of Γ, respectively. For each $y \in \mathbb{R}^m$, $\Gamma^{-1}(y) :=$

$\{x \in \mathbb{R}^n : y \in \Gamma(x)\}$ denotes the preimage of y. Γ is said to be a closed convex multifunction if its graph $\mathrm{Gr}(\Gamma)$ is a closed convex subset of $\mathbb{R}^n \times \mathbb{R}^m$. As suggested in Ref. [15], Γ_∞ is used to denote the multifunction whose graph is the recession cone of $\mathrm{Gr}(\Gamma)$.

Theorem 4.1 *Suppose that* $\Gamma\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *is a closed convex multifunction. If* $x_0 \in \mathrm{dom}(\Gamma)$ *and*

$$(\mathrm{dom}(\Gamma_\infty))^- = (\mathrm{dom}(\Gamma) - x_0)^\perp \tag{12}$$

then there exists $\tau > 0$ *such that*

$$\mathrm{dist}(y, \Gamma(x_0)) \le \tau\, \mathrm{dist}(x_0, \Gamma^{-1}(y)) \quad \textit{for all } y \in \mathbb{R}^m. \tag{13}$$

Proof Let $K := \mathrm{Gr}(\Gamma) - (x_0, 0)$ and define $A\colon \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ by $Ay := (0, y)$. Then $\Gamma(x_0) = A^{-1}(K)$ and $A^*\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ maps every $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}^m$ to y^*, because

$$\langle A^*(x^*, y^*), y\rangle = \langle (x^*, y^*), Ay\rangle = \langle (x^*, y^*), (0, y)\rangle = \langle y^*, y\rangle, \quad \forall\, y \in \mathbb{R}^m.$$

Therefore $\ker(A^*) = \mathbb{R}^n \times \{0\}$. We claim that $\ker(A^*) \cap \overline{\mathrm{barr}(K)} = \ker(A^*) \cap K^\perp$, that is,

$$(\mathbb{R}^n \times \{0\}) \cap \overline{\mathrm{barr}\,(\mathrm{Gr}(\Gamma) - (x_0, 0))} = (\mathbb{R}^n \times \{0\}) \cap (\mathrm{Gr}(\Gamma) - (x_0, 0))^\perp. \tag{14}$$

Granting this, by Theorem 3.1, we have for some $\tau > 0$,

$$\begin{aligned} \mathrm{dist}(y, \Gamma(x_0)) &= \mathrm{dist}(y, A^{-1}(K)) \le \tau\, \mathrm{dist}(Ay, K) = \tau\, \mathrm{dist}((x_0, y), \mathrm{Gr}(\Gamma)) \\ &\le \tau\, \mathrm{dist}\left((x_0, y), \Gamma^{-1}(y) \times \{y\}\right) = \tau\, \mathrm{dist}(x_0, \Gamma^{-1}(y)), \end{aligned}$$

where the second inequality holds because $\Gamma^{-1}(y) \times \{y\} \subset \mathrm{Gr}(\Gamma)$ and where the norm on $\mathbb{R}^n \times \mathbb{R}^m$ is the sum of the norm on $\mathbb{R}^n$ and the norm on $\mathbb{R}^m$.

Now we prove that (14) holds. It suffices to prove that

$$(\mathbb{R}^n \times \{0\}) \cap \overline{\mathrm{barr}(\mathrm{Gr}(\Gamma))} \subset (\mathbb{R}^n \times \{0\}) \cap (\mathrm{Gr}(\Gamma) - (x_0, 0))^\perp. \tag{15}$$

Let (x^*, y^*) belong to the left hand side of the above expression. Then $y^* = 0$ and $(x^*, 0) \in \overline{\mathrm{barr}(\mathrm{Gr}(\Gamma))}$. Since $(\mathrm{Gr}(\Gamma)_\infty)^- = \overline{\mathrm{barr}(\mathrm{Gr}(\Gamma))}$, $(x^*, 0) \in (\mathrm{Gr}(\Gamma)_\infty)^-$. It follows from the definition of the multifunction Γ_∞ that $(x^*, 0) \in (\mathrm{Gr}(\Gamma_\infty))^-$, which implies that $x^* \in (\mathrm{dom}(\Gamma_\infty))^-$. In view of (12), $x^* \in (\mathrm{dom}(\Gamma) - x_0)^\perp$. Therefore, for every $(x, y) \in \mathrm{Gr}(\Gamma)$,

$$\langle (x^*, y^*), (x, y) - (x_0, 0)\rangle = \langle (x^*, 0), (x, y) - (x_0, 0)\rangle = \langle x^*, x - x_0\rangle = 0,$$

i.e., $(x^*, y^*) \in (\mathrm{Gr}(\Gamma) - (x_0, 0))^\perp$. This verifies (15). □

Acknowledgments The author is grateful to the referees for valuable suggestions. One of them pointed out an incorrect usage of a property of recession cone in the previous version of this paper. This work was partially supported by Natural Science Foundation of China, Sichuan Youth Science and Technology Foundation (06ZQ026-013), and SZD0406 from Sichuan Province.

References

1. Burke, J.V., Tseng, P.: A unified analysis of Hoffman's bound via Fenchel duality. SIAM J. Optim. **6**(2), 265–282 (1996)
2. Gwinner, J.: Closed images of convex multivalued mappings in linear topological spaces with applications. J. Math. Anal. Appl. **60**(1), 75–86 (1977)
3. He, Y.R., Sun, J.: Error bounds for degenerate cone inclusion problems. Math. Oper. Res. **32**(3), 701–717 (2005)
4. He, Y.R., Sun, J.: Second order sufficient conditions for error bounds in Banach spaces. SIAM J. Optim. **17**(3), 795–805 (2006)
5. Hoffman, A.J.: On approximate solutions of systems of linear inequalities. J. Res. Nat. Bur. Stand. **49**, 263–265 (1952)
6. Huang, L.R., Ng, K.F.: On first- and second-order conditions for error bounds. SIAM J. Optim. **14**, 1057–1073 (2004)
7. Ioffe, A.D.: Regular points of Lipschitz functions. Trans. Am. Math. Soc. **251**, 61–69 (1979)
8. Lewis, A.S., Pang, J.-S.: Error bounds for convex inequality systems. In: Crouzeix, J.-P., Martinez-Legaz, J.-E. , Volle, M. (eds.) Generalized Convexity, Generalized Monotonicity: Recent Results., pp. 75–110. Kluwer, Dordrecht (1998)
9. Luenberger, D.G.: Optimization by Vector Space Methods. Wiley, New York (1969)
10. Ng, K.F., Yang, W.H.: Error bounds for abstract linear inequality systems. SIAM J. Optim. **13**(1), 24–43 (2002)
11. Ng, K.F., Zheng, X.Y.: Global error bounds with fractional exponents. Math. Program. Ser. B **88**(2), 357–370 (2000)
12. Rockafellar R.T.: Convex Analysis. Princeton University Press, Princeton, NJ (1970)
13. Wu, Z., Ye, J.J.: On error bounds for lower semicontinuous functions. Math. Program. Ser. A **92**(2), 301–314 (2002)
14. Zălinescu, C.: Convex Analysis in General Vector Spaces. World Scientific Publishing, River Edge, NJ (2002)
15. Zălinescu, C.: A nonlinear extension of Hoffman's error bound for linear inequalities. Math. Oper. Res. **28**(3), 524–532 (2003)